CONTROL OF CONVECTIVE FLOWS

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Outline of the lecture

- Introduction to flow control
- The Thermal Convection Loop
- The Rayleigh Bénrad Problem
- Outlook

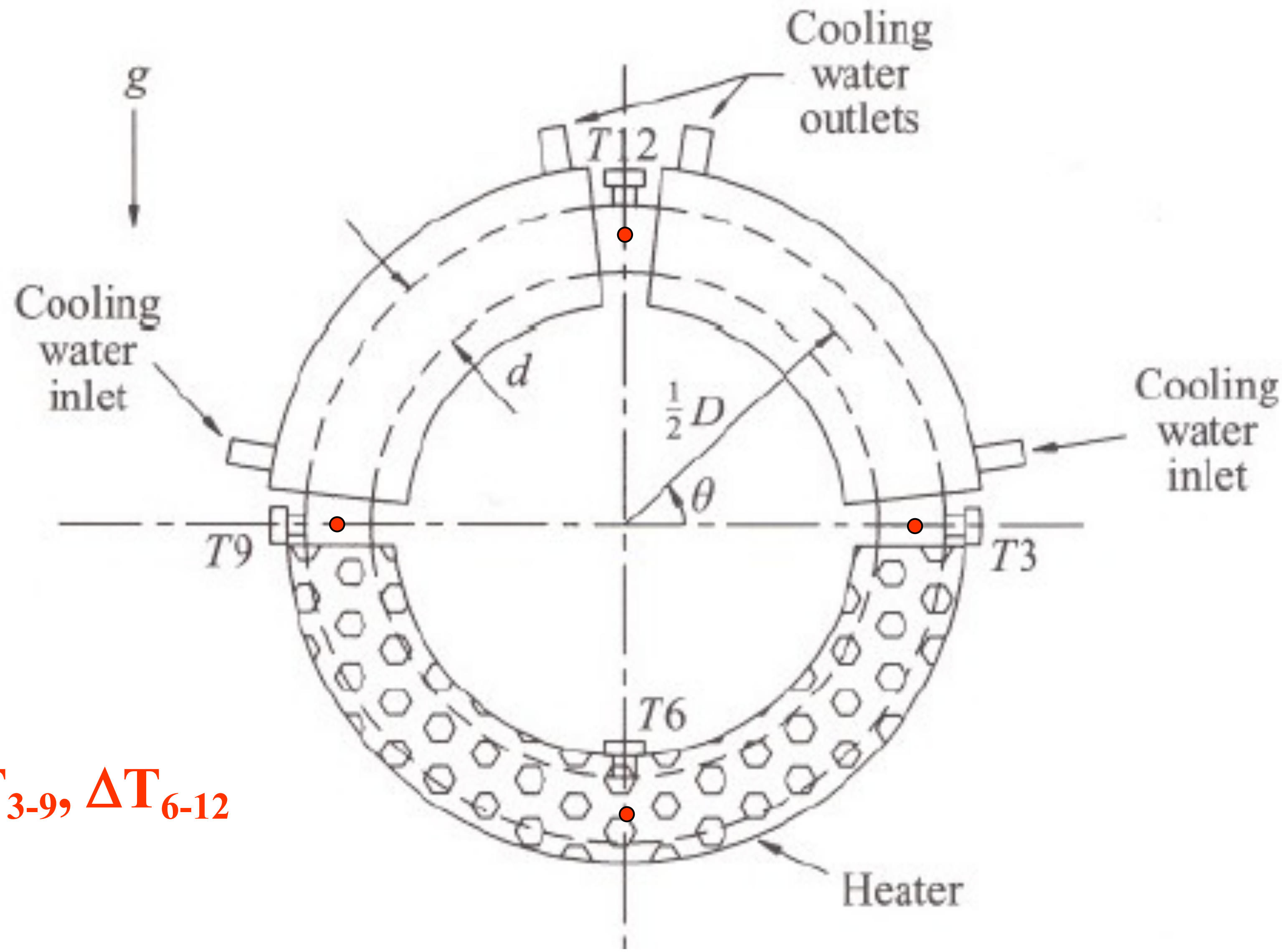
Introduction

- The ability to control flow patterns is important from both the technological and the scientific points of view:
  - The ability to control may lead to a deeper understanding of dynamic processes, i.e., the structure of strange attractors
  - Many industrial processes do not operate under optimal conditions. Control would enable to achieve better performance at a reduced cost
- Controller’s objectives:
  - Maintain flow conditions other than the naturally occurring ones
  - Stabilize otherwise non stable flows
  - Induce chaos/mixing in otherwise laminar flows
- The flow control problem is difficult
  - High dimension systems
  - Nonlinearities

Some applications of flow control

- Turbulence control: drag reduction in aircraft, ships, and submarines
- Flow control in turbo-machinery
- Crystal growth: suppression the convection and/or flow instability
- Aluminum production: suppression of interfacial instabilities (Davidson estimates that a 0.5cm reduction in the electrolyte thickness may lead to annual savings of over US $\$10^8$.)
- Chemical and biological reactions, combustion, and heat exchange: enhanced mixing

The Thermal Convection Loop



$\Delta T_{3\text{-}9}, \Delta T_{6\text{-}12}$

Singer, J., Wang, Y., Z., and Bau, H., H., 1991, Controlling a Chaotic System, ***Physical Review Letters***, 66, 1123-1125.

Wang Y., Singer J., and Bau, H. H., 1992, Controlling Chaos in a Thermal Convection Loop, ***J. Fluid Mechanics***, 237, 479-498.

A one-dimensional model for the thermal loop

- The continuity equation (Boussinesq's approximation):

$$u = u(t)$$

- The momentum equation:

$$\dot{u} = \frac{1}{\pi} Ra\, P \oint T \cos(\theta)\, d\theta - Pu$$

- The energy equation:

$$\dot{T} = -u \frac{\partial T}{\partial \theta} + B \frac{\partial^2 T}{\partial \theta^2} + \left[T_w(\theta, t) - T \right]$$

SPECTRAL EXPANSION

$$T_w(\theta,t) = W_0(t) + \sum_{n-1}^{\infty} W_n(t)\sin(n\theta),$$ (Wall temperature)

$$T(\theta,t) = \sum_{n=0}^{\infty} S_n(t)\sin(n\theta) + C_n(t)\cos(n\theta).$$ (Fluid's temperature)

- An infinite set of ODEs
- Three of the equations decouple from the rest of the set with exact closure:

$$\dot{x}_1 = P(x_2 - x_1)$$

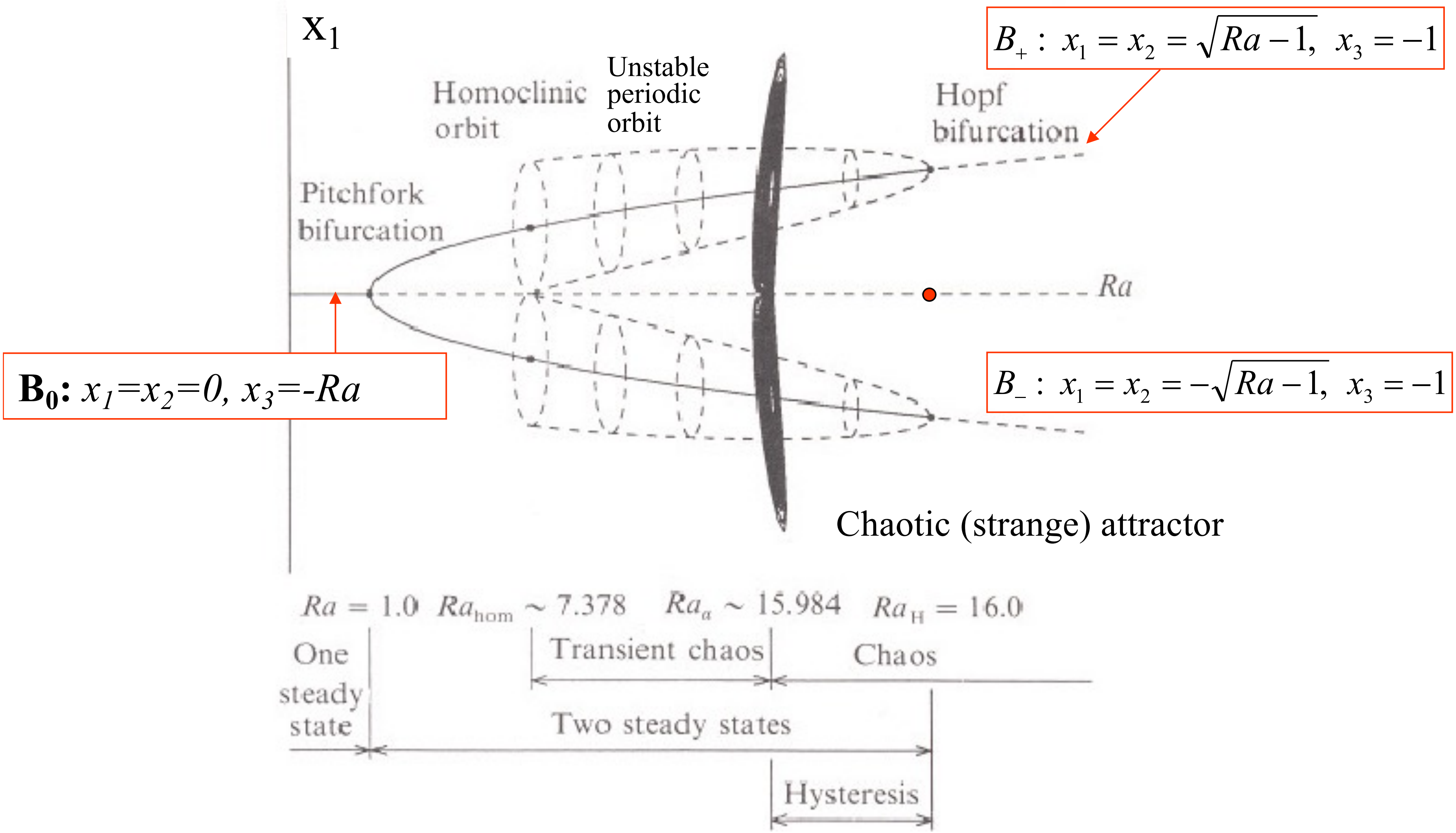
$$\dot{x}_2 = -x_1 x_3 - x_2$$

$$\dot{x}_3 = x_1 x_2 - x_3 + RaW_1$$

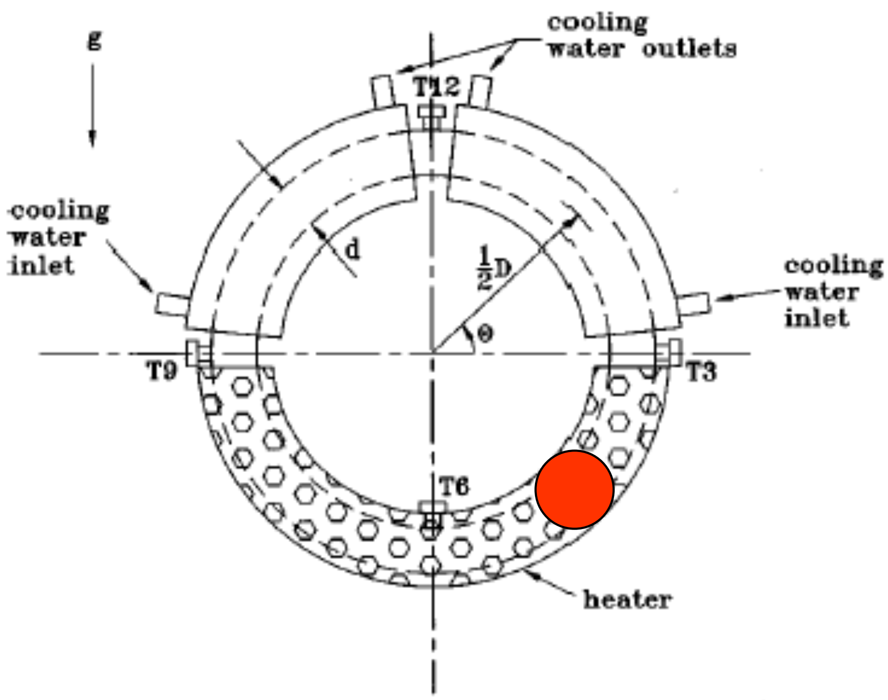
$x_1(t)$ is cross-sectionally averaged speed;

$x_2(t)$ and $x_3(t)$ are, respectively, proportional to the fluid's temperature differences between positions 3 and 9 o'clock and positions 12 and 6 o'clock around the loop

The equilibrium states of the uncontrolled loop (W=-1)



Physical mechanism for Flow Instabilities



Small disturbance causes a fluid parcel in the heater to be hotter than usual

↓

Buoyancy ⍰ , Velocity ⍰

↓

Fluid goes through the heater/cooler faster than usual

↓

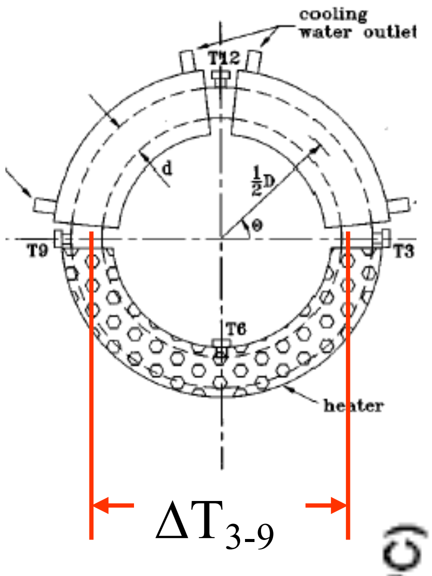
Buoyancy ⍰ , Velocity ⍰

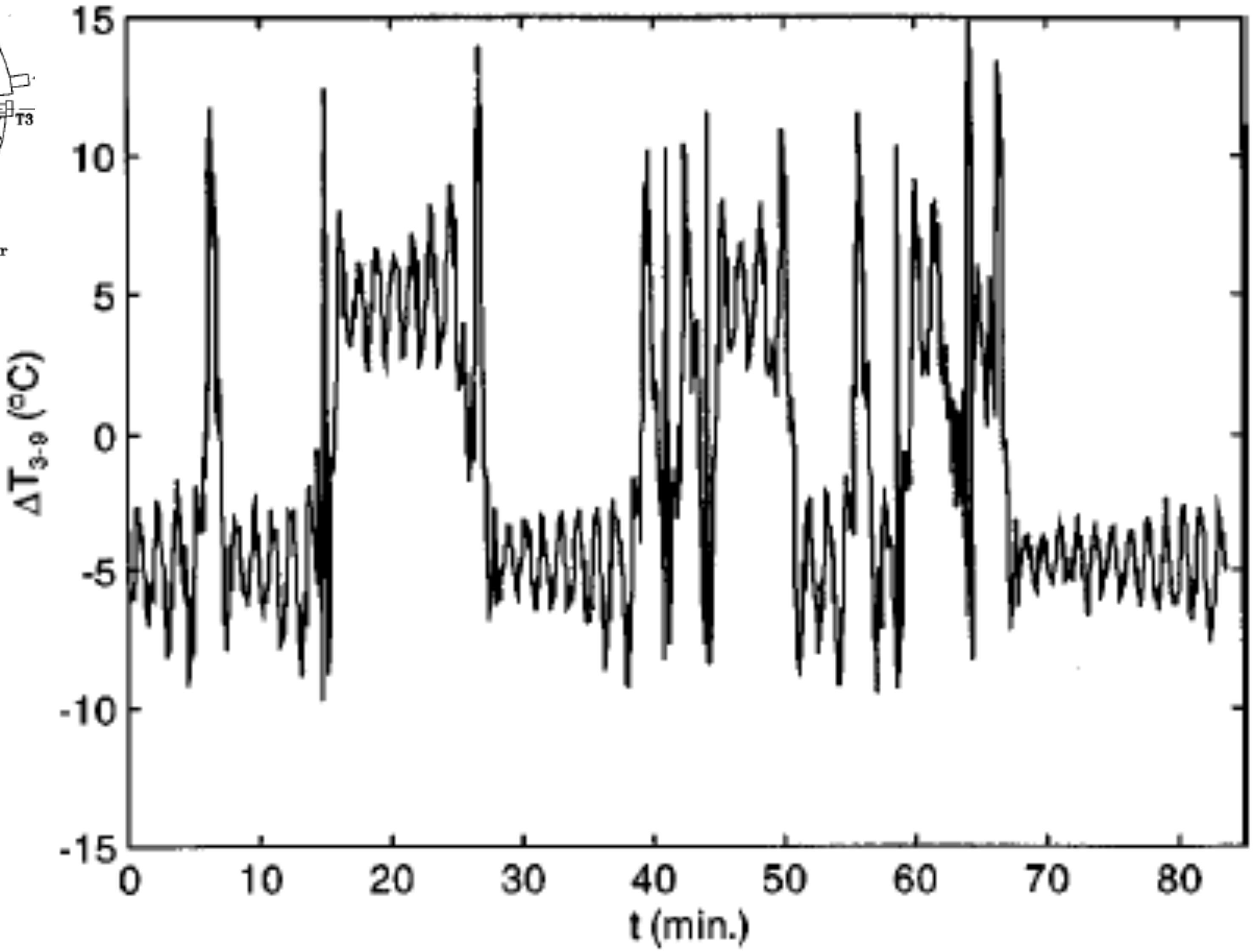
↓

Fluid goes through the heater/cooler slower than usual

(loops back to "Buoyancy ⍰ , Velocity ⍰")

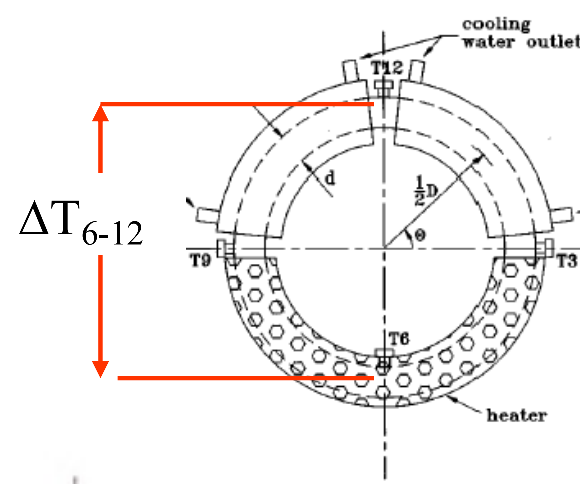
CHAOTIC ADVECTION

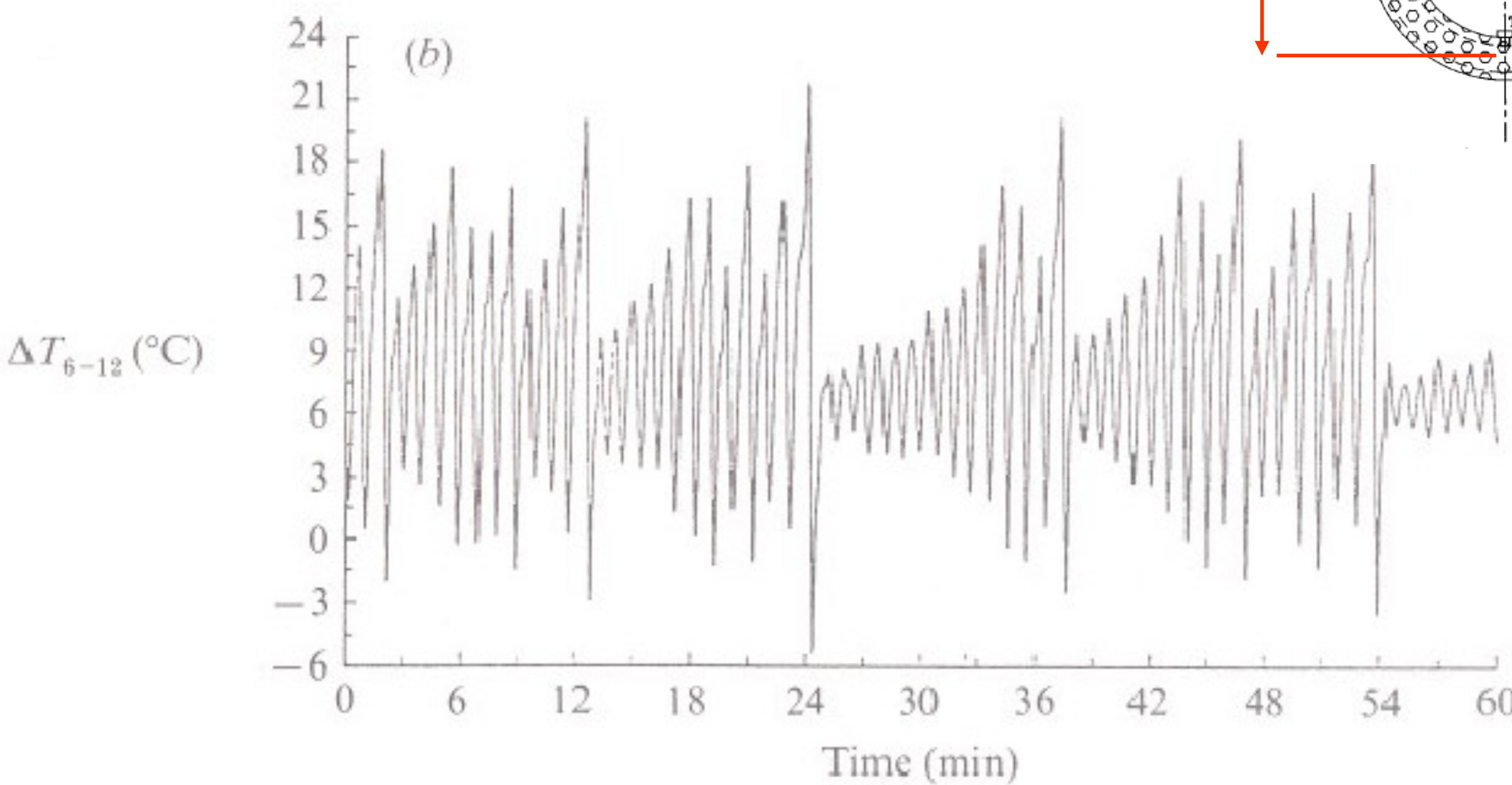




The experimentally observed temperature difference between positions 3 and 9 o'clock (ΔT_{3-9}) is depicted as a function of time. Ra=3Ra (Q=3Q_C).

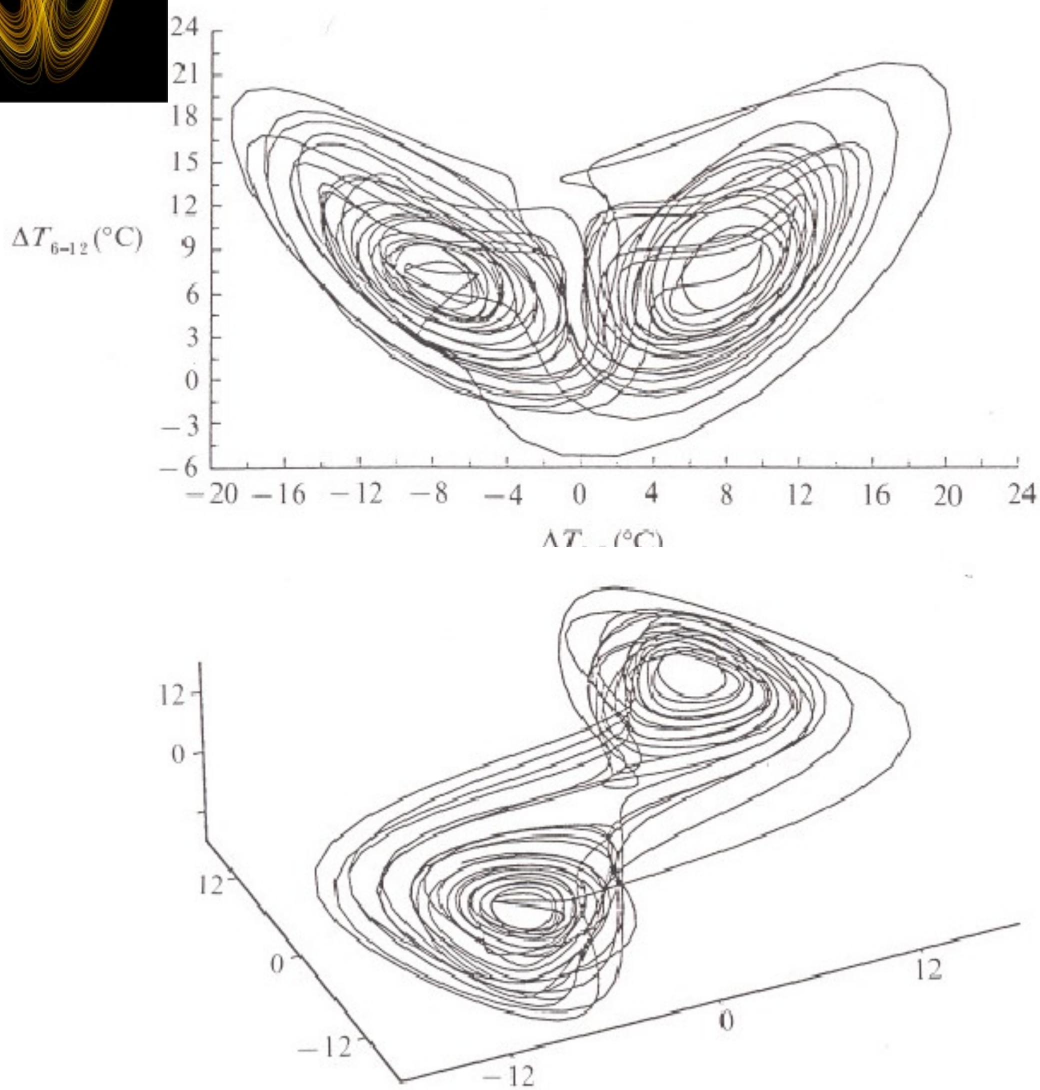
CHAOTIC ADVECTION



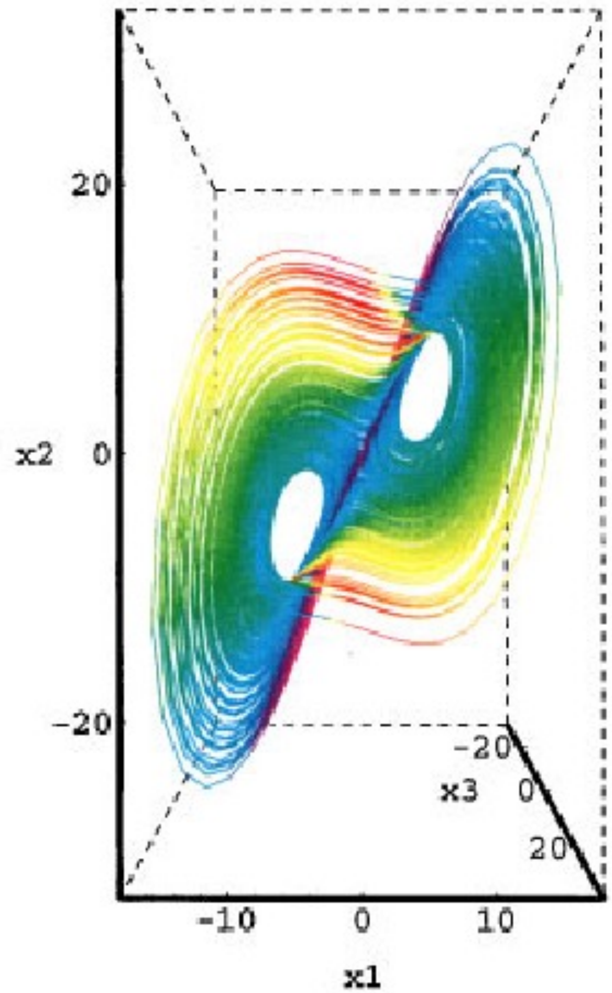


The experimentally measured temperature difference between positions 6 and 12 o'clock (ΔT_{6-12}) is depicted as a function of time. Ra=3Ra_H (Q=3Q_C).

PHASE-SPACE PORTRAITS



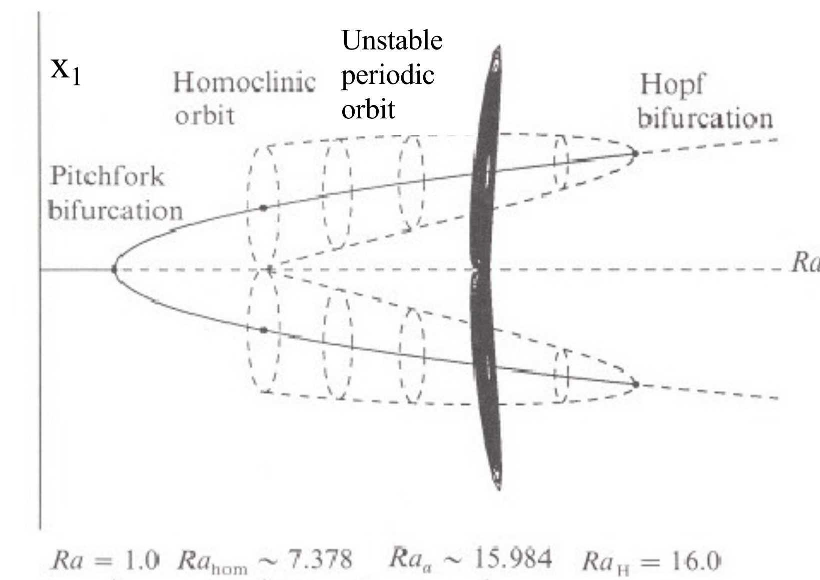
Reconstruction of the attractor from experimental data Ra=3Ra_H



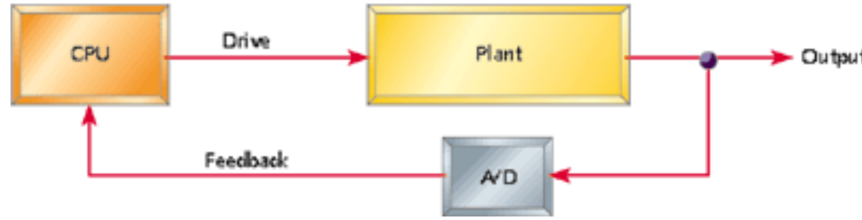
Phase portrait based on the solution of the Lorenz equations (Bewley, 1999)

Flow control objectives

- Maintain motionless state (B_0) when the motionless state is normally unstable (Ra>1) (Singer et al., 1991a).
- Maintain time-independent convections when $Ra>Ra_H$ *i.e.,suppress chaos* (Singer et al., 1991a, Wang et al., 1992, Burns et. al., 1998, Yuen & Bau, 1999).
- Maintain periodic motion of desired periodicity under conditions when the system would normally assume chaotic behavior. *There is an infinite number of non-stable chaotic orbits embedded within the chaotic attractor.* (Singer & Bau, 1991b, and Yuen & Bau, 1996).
- Induce chaos in otherwise laminar (fully predictable, $Ra<Ra_H$), non-chaotic flow (Wang et al, 1992).



Flow control strategies



Open loop control

- Appropriate design to achieve desired outcomes, *i.e.*, tilt the loop or provide asymmetric heating to suppress chaotic advection.
- Use pre-determined actuation, *i.e.*, modulate the heating rate as a function of time – periodic modulation of the heating rate delays transition from the no-motion to the convective state.

Closed loop (feedback) control

- Modulate the control input as a function of measured (observed) events in the plant, *i.e.*, modulate the heating rate as a function of the deviation of the measured temperature from a desired value. to enhance the disturbance-dissipating mechanisms and, in turn, stabilize the flow.

Feedback Control Strategies

- Ad-hoc, linear proportional (PID) control (Physically intuitive)
- Linear Quadratic Gaussian (LQG) Optimal control H_2: minimize a quadratic cost function (requires full knowledge of the plant's state)
- Linear robust control (H_∞): minimize a cost function subjected to the worst possible disturbances
- Nonlinear control
  - Linearizing controller
  - Neural networks
  - Polynomial controller to alter the direction of the bifurcation

Thermal loop control: ad-hoc, linear proportional controller

- Control objective

Suppress the chaotic behavior and maintain "laminar" flow

- Control strategy

Linear proportional feedback control

- The control model

Rewrite the equations in local form about the B_+ state

$$\mathbf{x} = \bar{\mathbf{x}} + \mathbf{x}'$$

$$\dot{x}' = f_x(Ra, x') + NL(x')$$

Local form about the B_+ state

$$\dot{\mathbf{x}}' = f(\mathbf{x}', \mathbf{u}) = A\mathbf{x}' + B\mathbf{u} + NL(\mathbf{x}') + G\boldsymbol{\zeta}$$

$$\mathbf{A} = \begin{pmatrix} -4 & 4 & 0 \\ -\bar{x}_3 & -1 & -\bar{x}_1 \\ \bar{x}_2 & \bar{x}_1 & -1 \end{pmatrix}$$

$$\mathbf{B}^T = \{0, 0, -1\}$$

$$\mathbf{NL}^T = \{0, -x'_1 x'_3, x'_1 x'_2\}$$

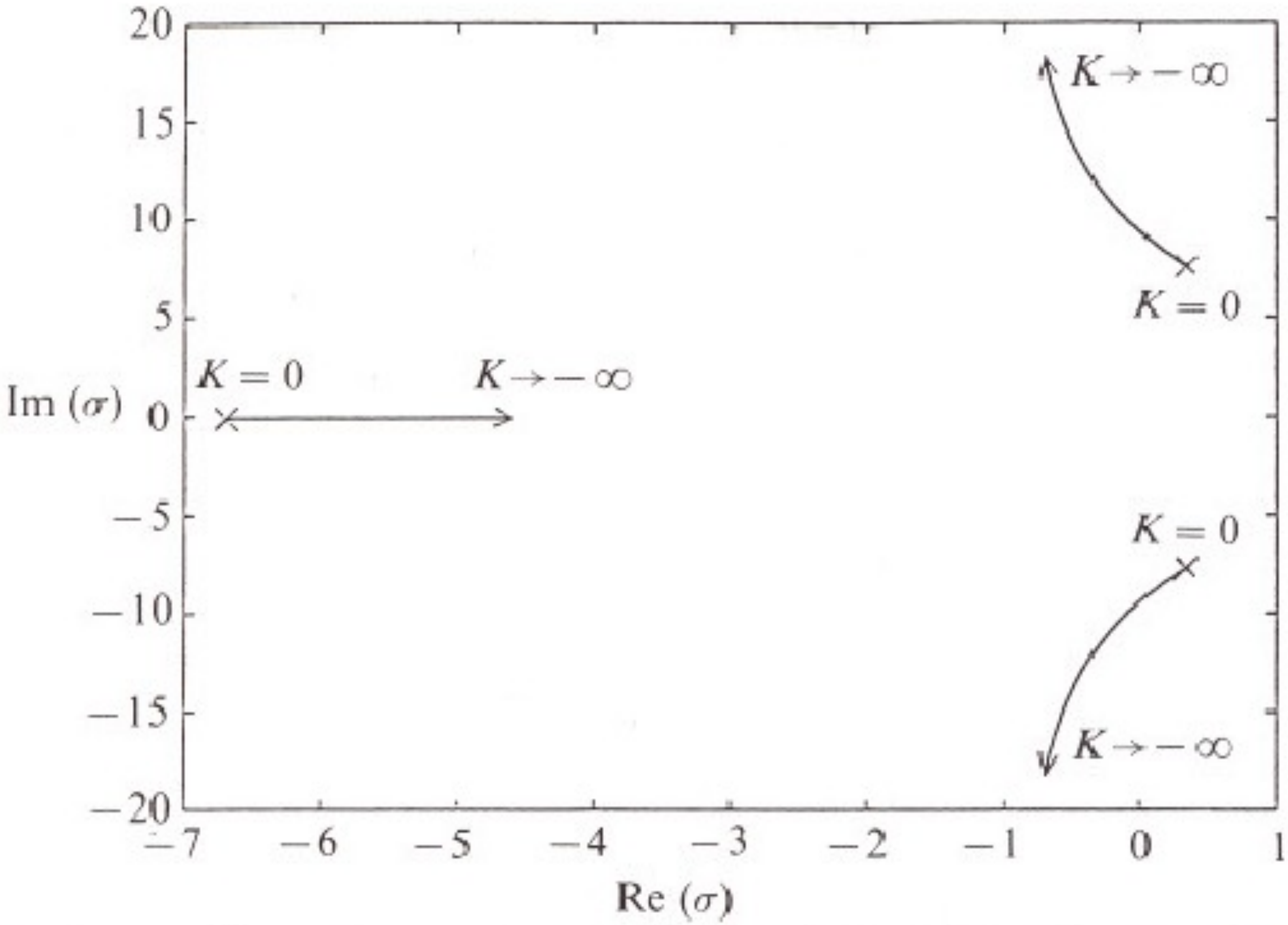
$$\mathbf{G}^T = \{0, 1, 0\}$$

Linear proportional controller: $u = Kx_2$

Linear stability analysis of the controlled state:

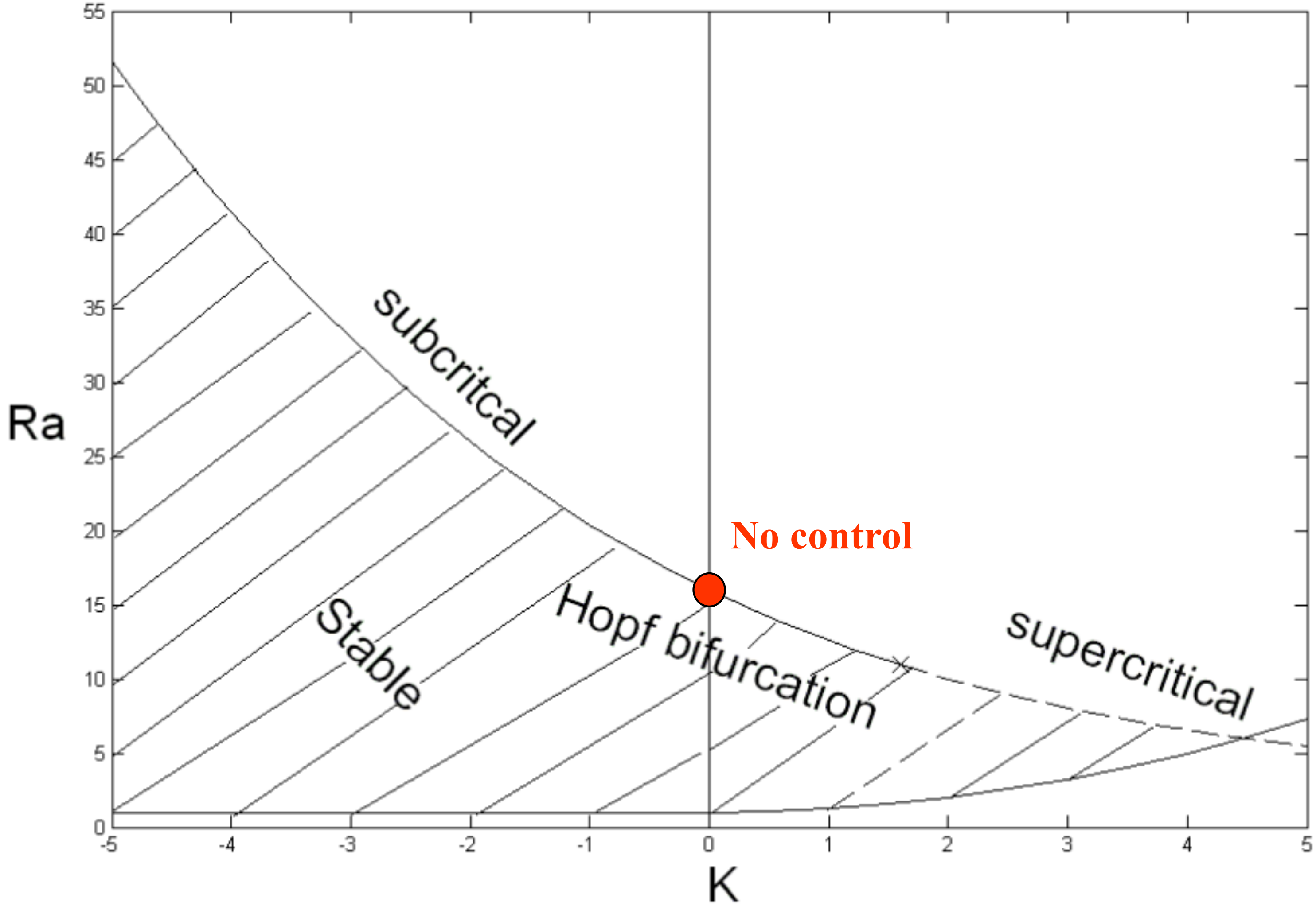
$$2\sqrt{Ra - 1} \geq K \qquad \frac{(P-2)(Ra - Ra_H(P))}{2\sqrt{Ra-1}} \leq -K$$

The controller affects the position of the eigenvalues of the controlled system



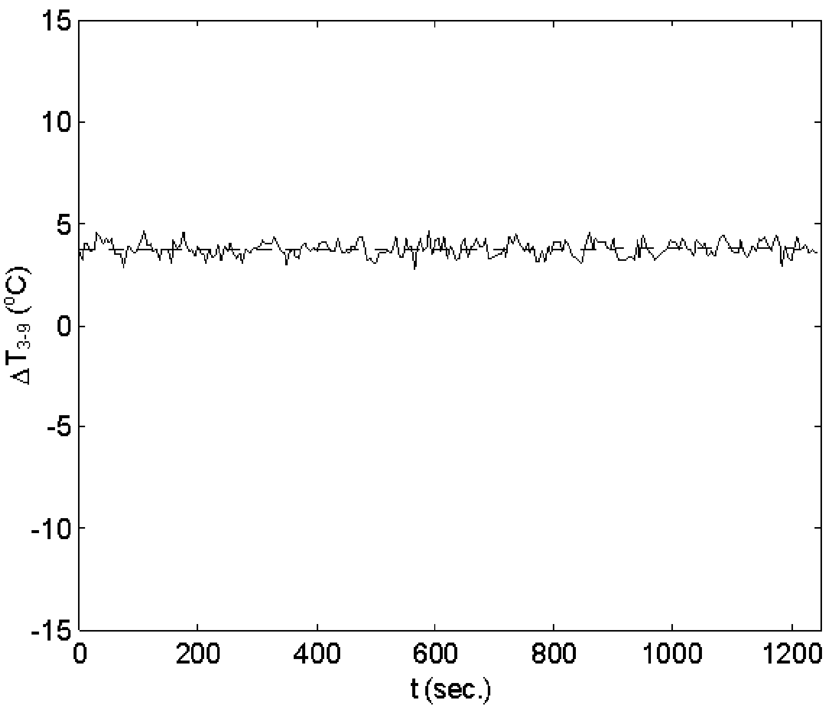
The eigenvalues of the controlled system depicted in the complex plane as functions of the proportional controller gain. K<0, Ra=50, and P=4

The critical Ra number as a function of the controller gain K. Stabilization of the B_+ state. P=4

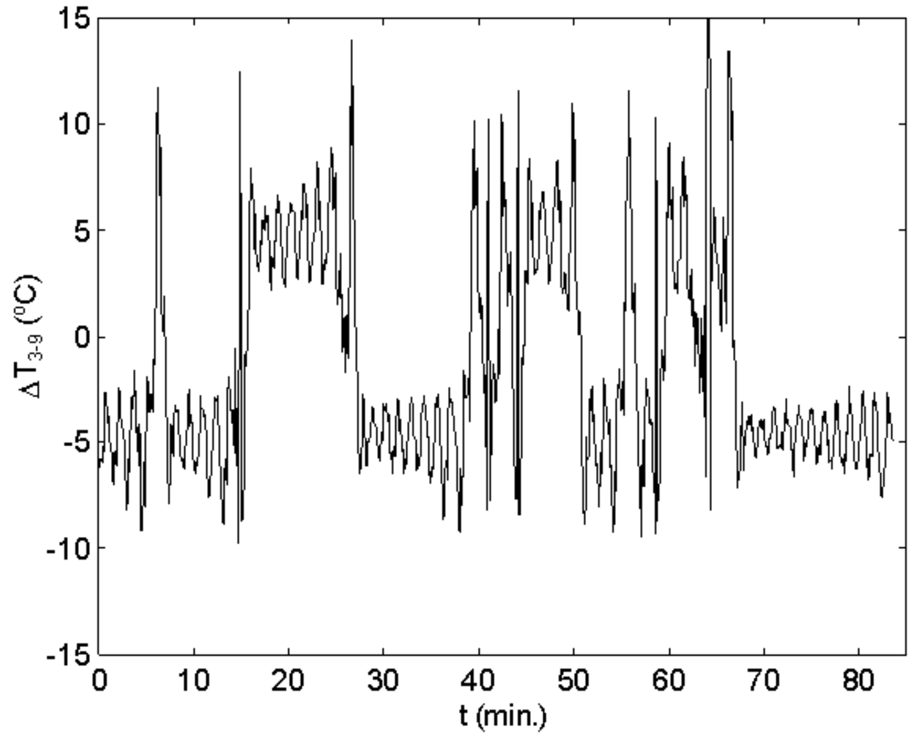


The temperature difference between positions 3 and 9 o'clock as a function of time

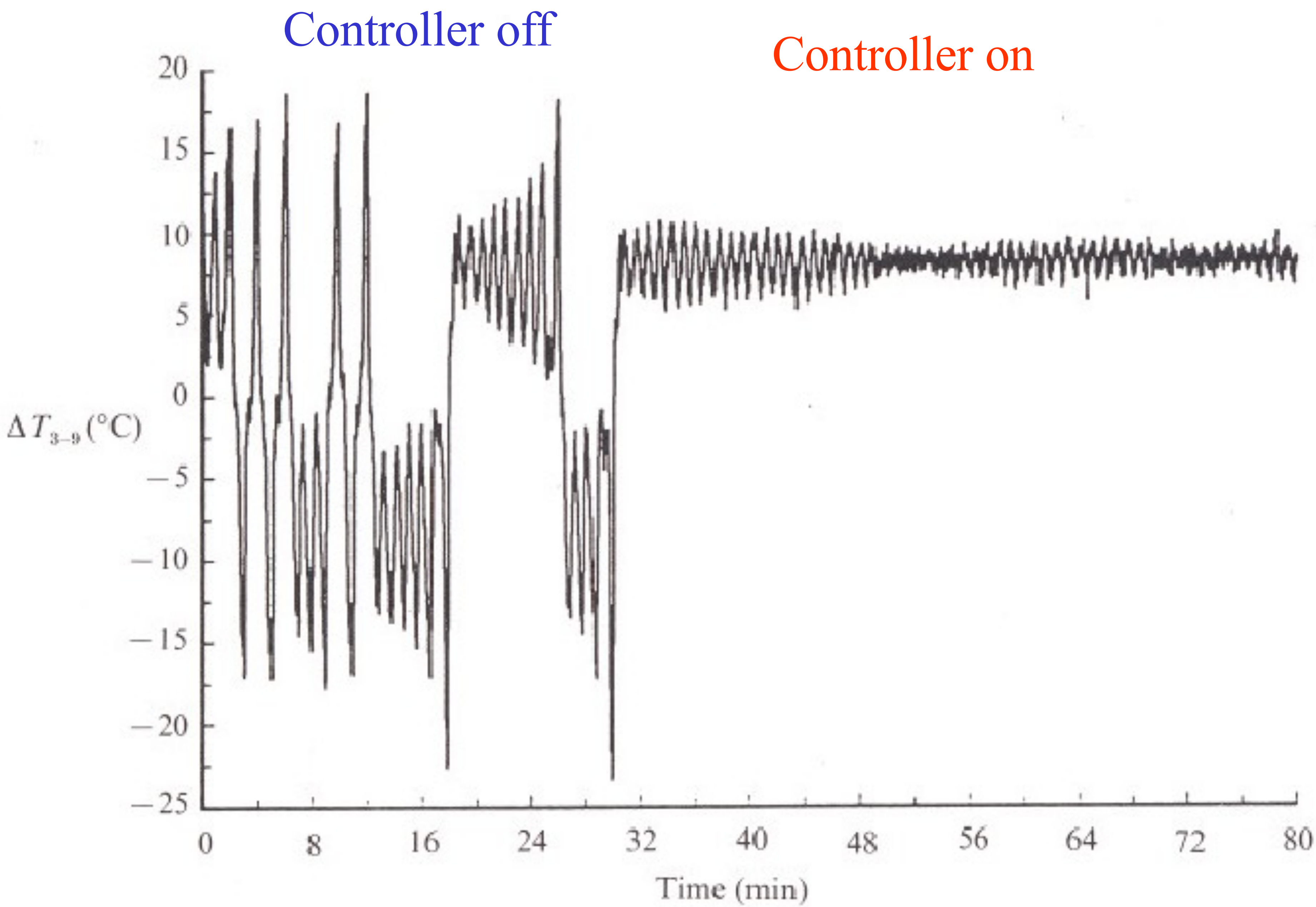
- **With proportional controller**



- **Without controller**

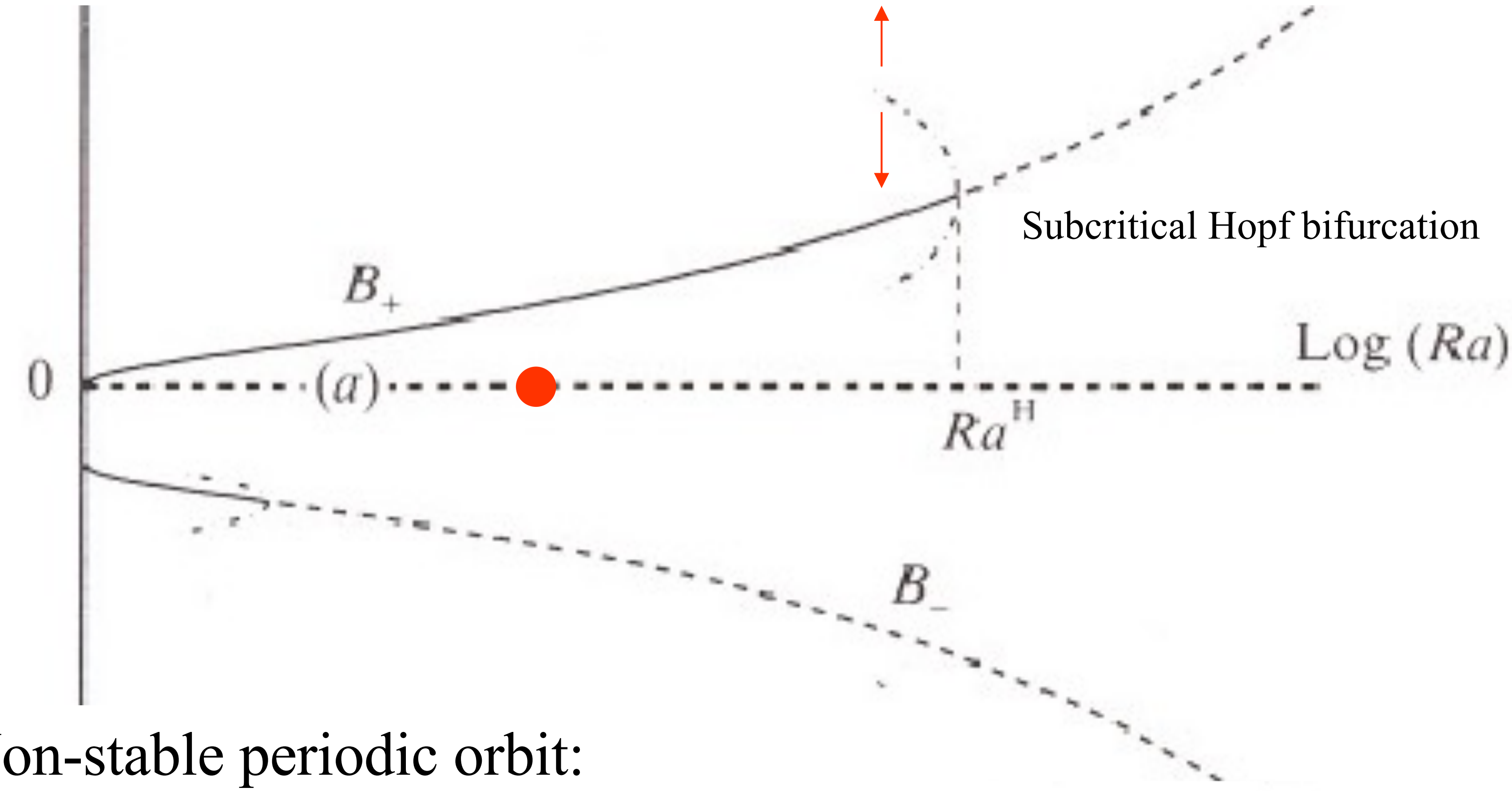


$$Ra=3Ra_C$$



Controller off
Controller on
20
15
10
5
0
−5
−10
−15
−20
−25
ΔT_{3-9} (°C)
0
8
16
24
32
40
48
56
64
72
80
Time (min)

BIFURCATION DIAGRAM OF THE CONTROLLED SYSTEM



Non-stable periodic orbit:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (Ra^{\mathrm{H}}(P,K)-1)^{\frac{1}{2}} \\ (Ra^{\mathrm{H}}(P,K)-1)^{\frac{1}{2}} \\ -1 \end{pmatrix} + 2\left[\frac{Ra - Ra^{\mathrm{H}}(P,K)}{\gamma(P,K)}\right]^{\frac{1}{2}} \begin{bmatrix} \cos[(\omega(P,K)t] \\ \cos[\omega(P,K)t] - (\omega_0/P)\sin[(\omega(P,K)t)] \\ A_1\cos[\omega(P,K)t] - A_2\sin[(\omega(P,K)t] \end{bmatrix}$$

Concern: the controlled system may have a limited basin of attraction

Nonlinear controller to invert the direction of the bifurcation

- In the presence of a subcritical bifurcation, the controlled state may have a limited basin of attraction
- The loss of stability may occur through non-linear bypass mechanisms
- This early transition would be less likely to occur for supercritical bifurcation.
- Nonlinear controllers may renders subcritical bifurcation supercritical and potentially increase the domain of attraction of the stable subcritical state

Yuen, P. K., & Bau, H. H., Rendering Subcritical Hopf Bifurcation Supercritical, ***J. Fluid Mechanics***, 317, 91-109, 1996.

Nonlinear controller on the thermal loop

- To implement the non-linear controller, we replace the control law with the nonlinear rule

$$u = k_p x_2(t) - k_n f\left(x_2(t)\right)$$

- $f(\chi)$ is a nonlinear function with $f(0)=f'(0)=0$ such as $f(\chi)=\chi 3$
- To avoid possible divergence of the controller $f(\chi)=\chi^3$, one can use a bounded function such as

$$f(\chi) = -3\left(\tanh(\chi) - \chi\right)$$

Weakly nonlinear analysis

- $\mathbf{x}$= {x1, x2, x3} denote, respectively, the deviations from $\mathbf{B}_+$, the local form will be:

$$L_1(Ra)\mathbf{x} = \dot{\mathbf{x}} + L_2(Ra)\mathbf{x} = \dot{\mathbf{x}} + \begin{pmatrix} 4 & -4 & 0 \\ -1 & 1 & \sqrt{Ra-1} \\ -\sqrt{Ra-1} & k_p - \sqrt{Ra-1} & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 - k_n x_2^{\,3} \end{pmatrix}$$

•Using a parametrization in terms of ε, expand $\boldsymbol{x}$ and Ra into the power series:

$$\mathbf{x} = \varepsilon\mathbf{x}_1 + \varepsilon^2\mathbf{x}_2 + \varepsilon^3\mathbf{x}_3 + \ldots c.c. = \varepsilon a(\tau_1, \tau_2, \ldots)\ \varsigma \exp(i\omega_0 t) + \varepsilon^2\mathbf{x}_2 + \varepsilon^3\mathbf{x}_3 + \ldots c.c.$$

$$Ra = Ra_H + \varepsilon^2 R_2 + \ldots$$

$\tau_j = \varepsilon^{2j} t$ are slow times;

Normalization: $[a\varepsilon, 1] = [\mathbf{x}, \varsigma^* e^{i\omega_0 t}]$

Leading order problem: $L_3\varsigma = [iw_0 I + L_2(Ra_H)]\varsigma = 0$

$$Ra_H = 1 + \frac{1}{4}\left(\sqrt{60 + k_p^{\,2}} - k_p\right)^2 \qquad \omega_0^{\,2} = Ra_H + 4 - k_p\sqrt{Ra_H - 1}$$

- The O(ε) equation is the linear stability problem
- At $O(\varepsilon^2)$:

$$L_1(Ra_H)\mathbf{x}_2 = \begin{pmatrix} 0 \\ -x_{1,1}x_{1,3} \\ x_{1,1}x_{1,2} \end{pmatrix}$$

- At $O(\varepsilon^3)$:

$$L_1(Ra_H)\mathbf{x}_3 = -\frac{\partial \mathbf{x}_1}{\partial \tau_1} + \frac{R_2}{2\sqrt{Ra_H - 1}} \begin{pmatrix} 0 \\ -x_{1,3} \\ x_{1,1} + x_{1,2} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{2,1}x_{1,3} - x_{1,1}x_{2,3} \\ x_{1,1}x_{2,2} + x_{2,1}x_{1,2} \end{pmatrix} - k_n \begin{pmatrix} 0 \\ 0 \\ x_{1,2}^{\;3} \end{pmatrix}$$

- impose solvability condition on the RHS of the $O(\varepsilon^3)$ equation to obtain the equation:

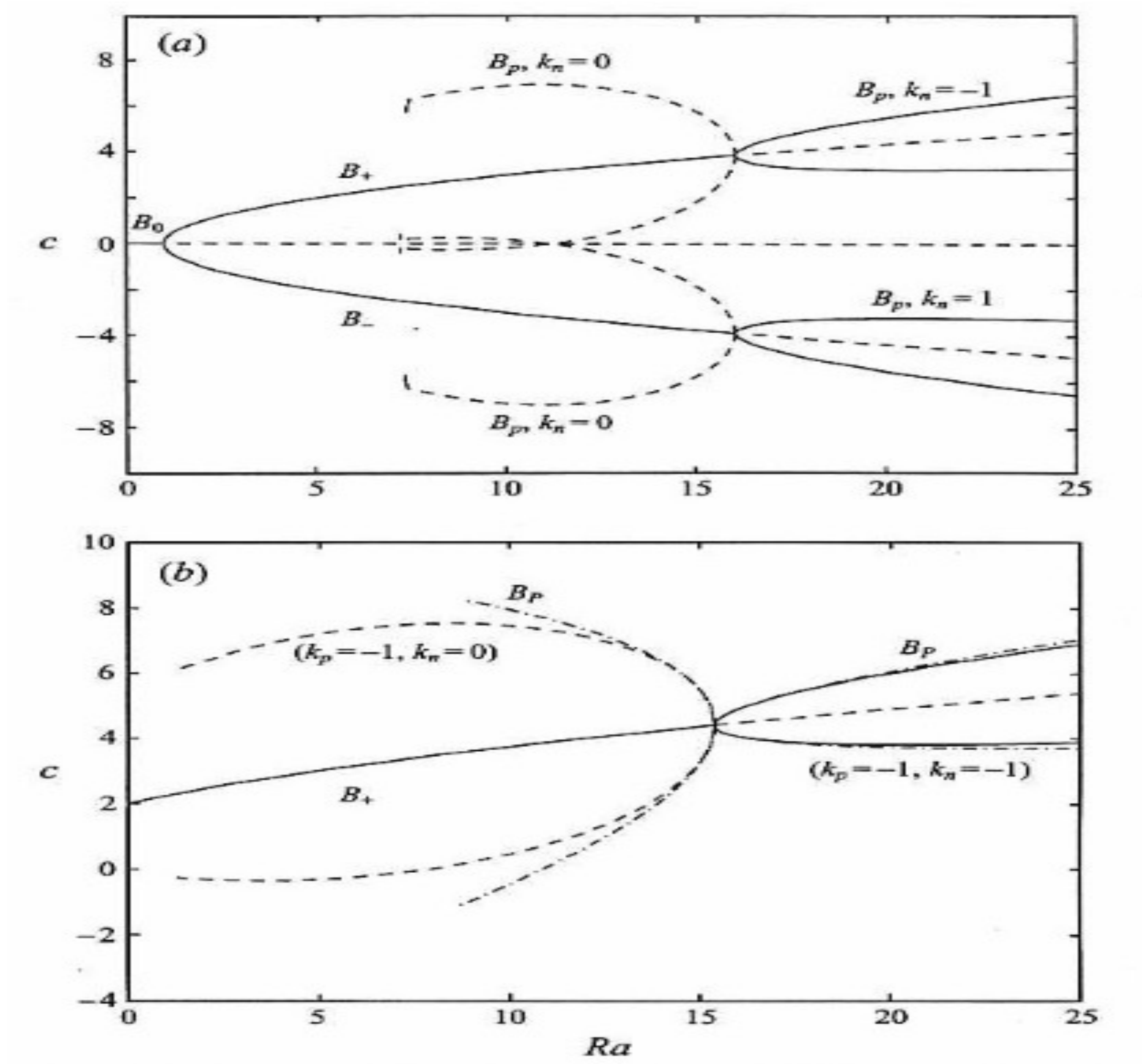
$$\frac{\partial a}{\partial \tau_1} = c_1 a \left\{ \left[c_2 |a|^2 (k_{n,c} + k_n) + R_2 \right] + i\left[-|a|^2 (c_3 + c_4 k_n) + c_5 R_2 \right] \right\}$$

- The amplitude equation:

$$\frac{\partial |a|^2}{\partial \tau_1} = 2c_1 |a|^2 \left[c_2 |a|^2 (k_{n,c} + k_n) + R_2 \right] \qquad |a|^2 = -\frac{R_2}{c_2 (k_{n,c} + k_n)}$$

The performance of nonlinear controller

The states $\mathbf{B}_+$ and $\mathbf{B}_\mathbf{P}$ (periodic orbit) are depicted as functions of *Ra* for $k_n=0$, $k_n=-1$, and $P=4$. (*a*) $k_p=0$, (*b*) $k_p=-1$. The solid and dashed lines correspond, respectively, to linearly stable and non-stable numerical solutions. The dash-dot line in (b) represents the analytic solution.



Introduction to optimal control

$$\dot{\mathbf{x}}' = f(\mathbf{x}', \mathbf{u}) = A\mathbf{x}' + B\mathbf{u} + NL(\mathbf{x}') + G\zeta$$

- Cost function:

$$J_{\mathbf{x}} = \frac{1}{2(t_1 - t_0)} \int_{t_0}^{t_1} \left(\mathbf{x}'^T \mathbf{Q}\mathbf{x}' + u^T R u \right) dt$$

- The optimal controller seeks a controller ***u*** that minimizes the cost function *J*. Q and R are weight functions
- To account for the system's constraints, we define the Hamiltonian:

$$\Xi = \lambda^T \left(-\dot{x}' + Ax' + Bu + NL(x') \right) - \frac{1}{2} \left(x'^T Qx' + u^T Ru \right)$$

- The Lagrange multipliers, λ(t), satisfy the adjoint equation:

$$\dot{\lambda} = -\left(\mathbf{A} + \frac{\partial \mathbf{NL}(\mathbf{x}')}{\partial \mathbf{x}'} \right)^T \lambda + \mathbf{Q}\mathbf{x}'$$

- The optimal controller is

$$u = R^{-1} \mathbf{B}^T \lambda$$

Problem: $\lambda(x_0, x, t)$

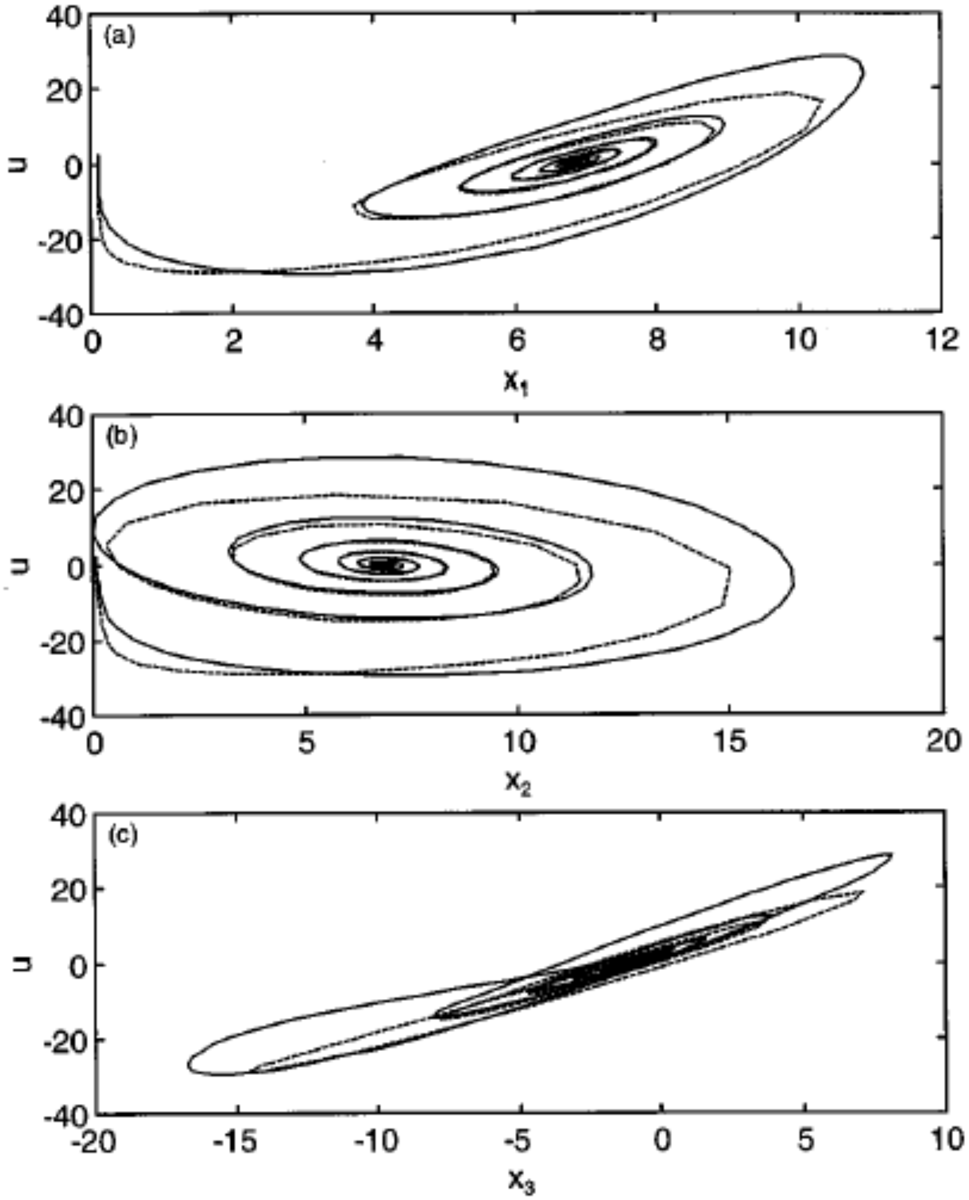


FIG. 3. The behavior of the nonlinear system subject to optimal control in the absence of stochastic noise. Ra$=$3 Ra$_H$$=$48. The control signal, u, is depicted as a function of $\mathbf{x}$. The solid and dashed lines represent, respectively, a nonlinear optimal controller and a linear, $\mathbf{K}_c=\{0.47, -0.54, 2.07\}$, optimal controller.

The nonlinear controller depends on:

Initial conditions (x'_0)

Current system's state variables (x')

and

Time (t)

Linear optimal controller

- Drop the nonlinear term **NL(x')**
- Assume that only x_2 is measured.

$$\dot{\mathbf{x}}' = A\mathbf{x}' + B\mathbf{u} + G\boldsymbol{\zeta}$$

$$\mathbf{y}_i = \mathbf{C}_i\mathbf{x}' + \mathbf{n}_i$$

$\mathbf{C}_1=\{0,1,0\}$, $\mathbf{C}_3=\mathbf{I}$, and **I** is the identity matrix and $\mathbf{n}_i(t)$ represents observation noise

- **Controllable**

$$rank[\mathbf{B}|\mathbf{AB}|\mathbf{A}^2\mathbf{B}] = 3$$

by a proper choice of the input *u*, one can transfer the plant from any state $\mathbf{x}'(t_0)$ at time $t=t_0$ to any other state, $\mathbf{x}'_1(t)$, in a finite time, $(t-t_0)$.

- **Observable**

$$rank[\mathbf{C}_i^T|\mathbf{A}^T\mathbf{C}_i^T|(\mathbf{A}^T)^2\mathbf{C}_i^T] = 3$$

given output **y** and the input *u* in the time interval $t_0<t<t_1$, one can deduct the initial state $\mathbf{x}'(t_0)$

Construction of the linear optimal regulator

- Minimize

$$J_{\mathbf{x}} = \frac{1}{2(t_1 - t_0)} \int_{t_0}^{t_1} \left(\mathbf{x}'^T \mathbf{Q} \mathbf{x}' + u^T R u \right) dt$$

Let $t_1 \rightarrow$ ⍰ to obtain the time-independent controller:

$$u = \mathbf{K}_c \mathbf{x}'$$

$$\mathbf{K}_c = -R^{-1} \mathbf{B}^T \mathbf{S}$$

S is the solution of the algebraic, matrix *Ricatti* equation

$$0 = \mathbf{S}\mathbf{A} + \mathbf{A}^T \mathbf{S} - \mathbf{S}\mathbf{B}R^{-1}\mathbf{B}^T \mathbf{S} + \mathbf{Q}$$

Algorithms for the solution of the Ricatti equation are available, i.e., Matlab

The basin of attraction of the linear controller (non-linear system)

- The linear controller guarantees that any disturbances will decay asymptotically (t→⍰) to the the controlled state.
- Often the linear operator of the controlled system may be non-normal and disturbances may not decay monotonically. In fact, even small disturbances may amplify a great deal prior to their eventual decay. Once amplified, the disturbances may trigger un- modeled dynamics. Hence, it is important to investigate the basin of attraction of the controlled state .
- The basin of attraction of the controlled state Γ_c is a region Ω_{BA} of phase space such that that

$$\lim_{x(t_0)\in\Omega_{BA}} x'(t) = 0$$

- In order to estimate the size of the *basin of attraction* of the controlled system, we construct a *Lyapunov function* or "*energy*"

Lyapunov function

- Lyapunov function: $H(x')>0$, $H(0)=0$ for all $x'\neq 0$, where $x'=0$ is the fixed point of the controlled system
- There is no systematic way to construct an “optimal” Lyapunov function for nonlinear systems.
- H satisfies a scalar equation of the form: $$\dot{H} = \frac{dH}{dt} = F(\mathbf{x}')$$
- $F(\mathbf{x}')=0$ divide the phase space into subspace (I) in which $\dot{H} < 0$ and subspace (II) in which $\dot{H} > 0$

An estimate of the basin of attraction

- Assume $H(\mathbf{x'})=H_1$ is the largest "hyper-sphere" that contains the origin (x'=0) and is fully contained in region (*I*).
- All trajectories starting inside H_1, eventually monotonically decay to zero.
- The "sphere" H_1 provides a lower bound (a conservative estimate) of the subspace of *monotonically* decaying disturbances.
- The size of the "sphere" H_1 depends sensitively on the choice of the Lyapunov function. H_1 ? Ω_{BA}

The H_2 sphere

- $H(\mathbf{x'}_0)>H_1$ do not necessarily diverge
  - ✓ Trajectories starting in subspace *II*, may eventually cross over to subspace *I*, and converge to the origin.
  - ✓ Trajectories starting in subspace *I* with $H(\mathbf{x'}_0)>H_1$ are not guaranteed to end up at the origin. Such trajectories may cross over to subspace *II*, and eventually end up on a different attractor.
- define a second "sphere," $H_2 \geq H_1$ such that for all $\mathbf{x'}_0$ when $H(\mathbf{x'}_0)<H_2$ and $t \to \infty$, $H \to 0$ and $\mathbf{x'}(t) \to 0$, albeit not necessarily monotonically.

A Lyapunov function of thermal loop system

- The controlled system:

$$\dot{\mathbf{x}}' = \mathbf{A_C}\mathbf{x}' + \mathbf{NL}(\mathbf{x}') \qquad \mathbf{A}_C = \begin{pmatrix} -4 & 4 & 0 \\ -\bar{x}_3 & -1 & -\bar{x}_1 \\ \bar{x}_2 - k_1 & \bar{x}_1 - k_2 & -1 - k_3 \end{pmatrix}$$

- Denote the eigenvalues and eigenvectors of $\mathbf{A_C}$ as $\{\eta_1, \eta_2 \pm i\eta_3\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2 \pm i\boldsymbol{v}_3\}$, where η_i and $\boldsymbol{v}_i$ are real.
- introduce the vector $\boldsymbol{z} = \boldsymbol{V}^{-1}\boldsymbol{x}'$, where $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$

$$\dot{\mathbf{z}} = \mathbf{V}^{-1}\mathbf{A_C}\mathbf{V}\mathbf{z} + \mathbf{V}^{-1}\mathbf{NL}(\mathbf{V}\mathbf{z})$$

- define the Lyapunov function:

$$H = \mathbf{z}^T \mathbf{C} \mathbf{z}$$

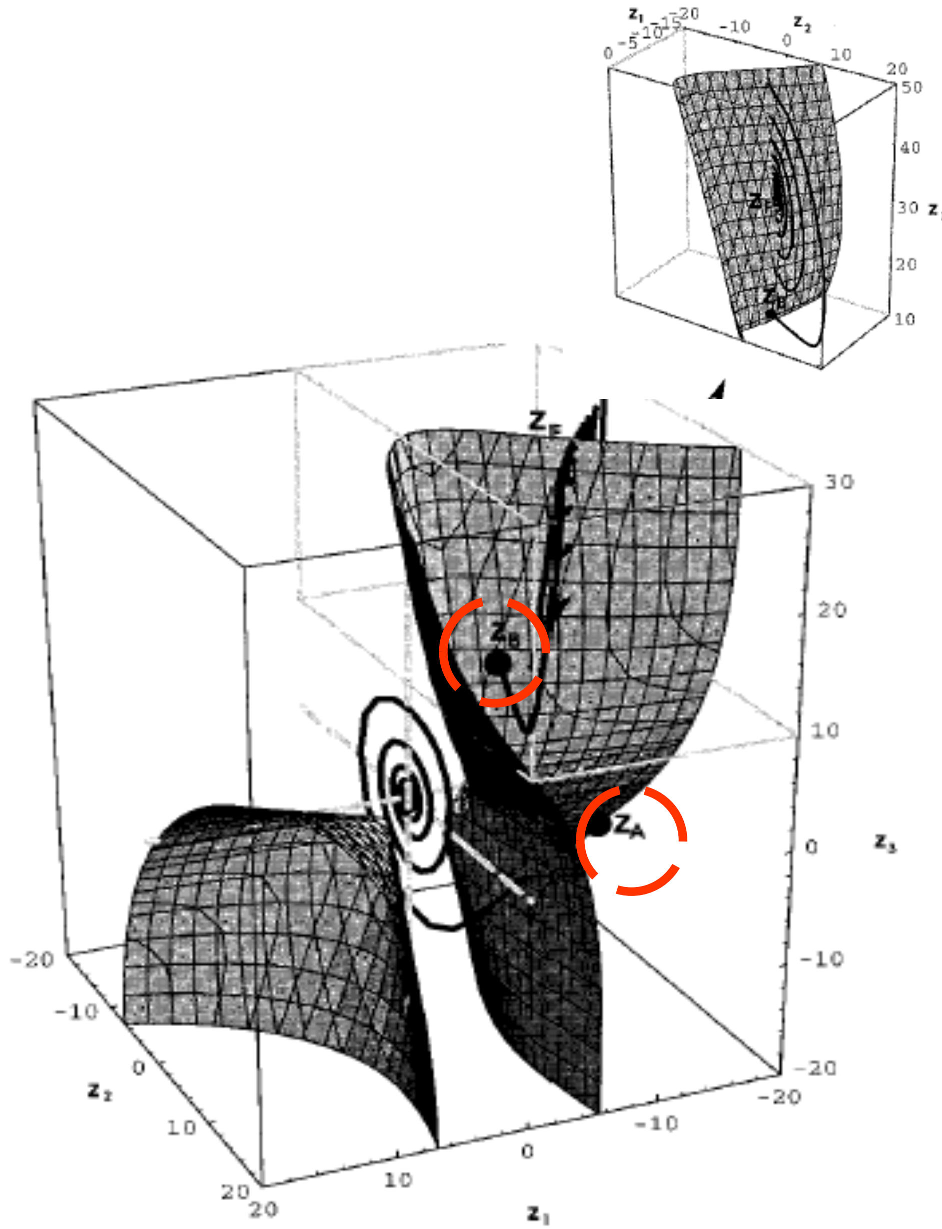
$$\begin{pmatrix} -\frac{1}{2\eta_1} & 0 & 0 \\ 0 & -\frac{1}{2\eta_2} & 0 \\ 0 & 0 & -\frac{1}{2\eta_2} \end{pmatrix}$$

.

$$\dot{H} = F(\mathbf{z}) = -\mathbf{z}^T \mathbf{z} + 2\mathbf{z}^T \mathbf{C} \mathbf{V}^{-1} NL(\mathbf{V}\mathbf{z})$$

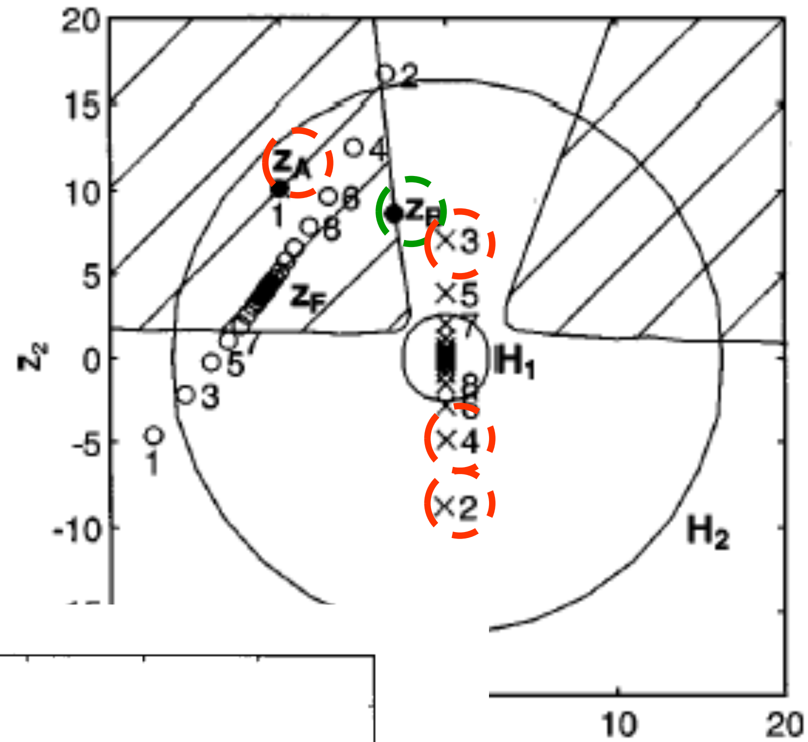
The basin of attraction of the linear controller

The surfaces are depicted as functions of the coordinates z_1, z_2, and z_3 in a three-dimensional phase space. The desired, set-state is at the origin. Trajectory *A* starting in subspace *II* (dH/dt>0) at $z=z_A$, where $H_1<H(z_A)<H_2$, is in the domain of attraction of z=0. Trajectory *B* starting in subspace II at $z=z_B$, where $H(z_B)>H_2$, is attracted to another fixed point, $z=z_F\neq 0$, of the controlled system. $Ra=3Ra_H=48$ and $Kc=\{0.47, -0.54, 2.07\}$.



The basin of attraction of the linear controller

- The phase space of the last slide is projected on the plane $z_3=0$. The spheres H_1 and H_2 are, respectively, conservative estimates of the domains of monotonic decay and the basin of attraction of the controlled state, $z=0$. The ×'s and o's represent, respectively, the penetration points of trajectory A starting at $z=z_A$ and trajectory B starting at $z=z_B$. The numbers next to the ×'s and o's denote the order of penetrations. The blank and shaded regions correspond, respectively, to subspaces I and *II*.



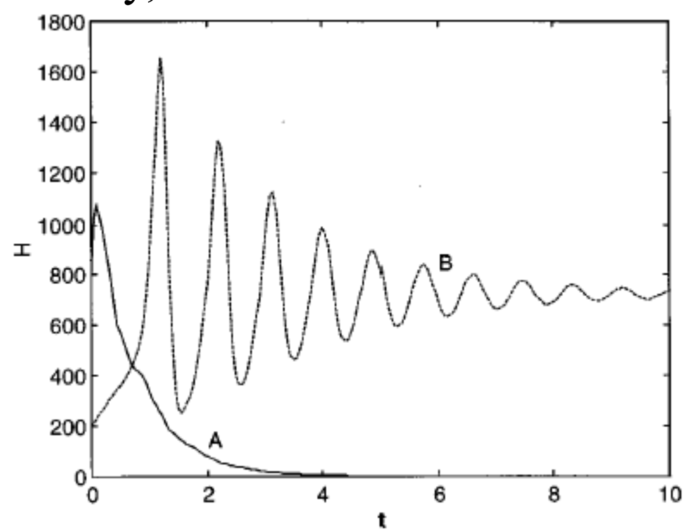


FIG. 6. The magnitudes of the Lyapunov functions, $H(t)$, associated with the trajectories A and B shown in Figs. 4 and 5 are depicted as functions of time.

The state estimator

- When the full state information is not available, or in the presence of measurement noise, an estimator needs to be constructed to estimate the state $\boldsymbol{x}'$ from the observed data $\boldsymbol{y}$.
- The controller will become: $u = \mathbf{K}_c \hat{\mathbf{x}}$
- In the above $\hat{\mathbf{x}}$ is the state estimate. The deviation between the estimate and the actual state should be as small as possible.

$$\mathbf{e}(t) = \mathbf{x}'(t) - \hat{\mathbf{x}}(t)$$

The nonlinear estimator

- We can construct the following dynamic system for the estimator

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_f\left(\mathbf{y}_i - \mathbf{C}_i\hat{\mathbf{x}}\right) + \mathbf{NL}(\hat{\mathbf{x}})$$

K_f is known as the estimator's filter

- The corresponding error equation is:

$$\frac{d\mathbf{e}}{dt} = \mathbf{A}^*\mathbf{e} - \mathbf{NL}(\mathbf{e}) + \left(\mathbf{G}\zeta - \mathbf{K}_f\mathbf{n}_i\right) \qquad \mathbf{A}^* = \left(\mathbf{A} + \frac{\partial \mathbf{NL}(\mathbf{x}')}{\partial \mathbf{x}'} - \mathbf{K}_f\mathbf{C}_i\right)$$

- The estimator tries to minimize
-

$$E\left(\int_{t_0}^{t_1} \mathbf{e}^T\mathbf{e}\,dt\right)$$

- Unfortunately, A^* requires knowledge of the state, $\mathbf{x}'$ which is not available

The filter for the nonlinear estimator

- Instead of minimizing $E\left(\int_{t_0}^{t_1} \mathbf{e}^T \mathbf{e}\, dt\right)$, as a more modest objective, we determine the filter $\boldsymbol{K}_f$ that renders the state $\boldsymbol{e}=0$ locally attractive
- Local attraction is guaranteed when the logarithmic norm

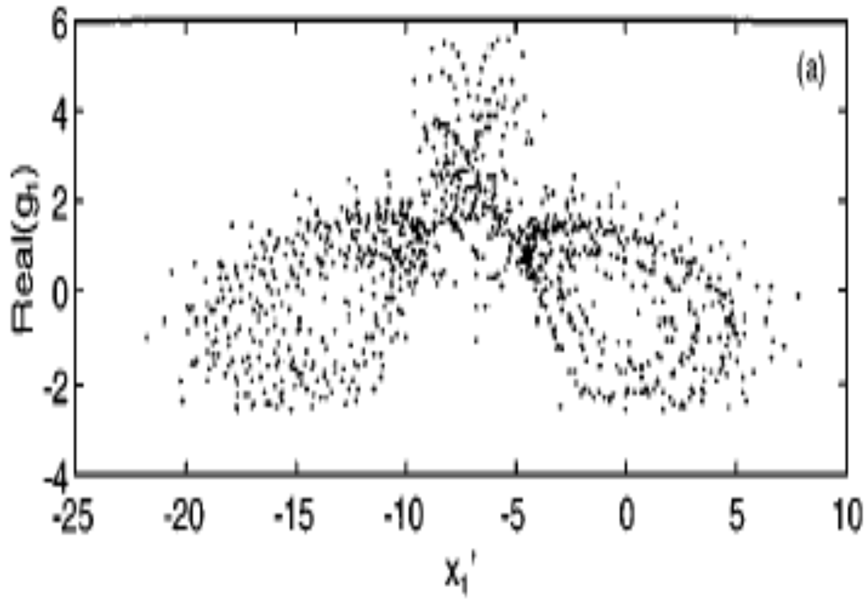
$$\mu_\infty\left(\mathbf{A}^*\right) = \max_i \left(A^*_{i,i} + \sum_{j,j\neq i} \left| A^*_{i,j} \right| \right)$$

is negative definite

- The logarithmic norm being negative is a conservative requirement that is sufficient, but not necessary, to assure that $\boldsymbol{e}=0$ is attractive

The nonlinear estimator's effect on thermal loop

- The largest real part of the state-dependent eigenvalues, $Real(g_1)$, is depicted as a function of **x**'. $Ra=3RaH=48$ and $\boldsymbol{K}_f^T = \{0.36, 1.75, -0.35\}$.
- $g_i(\boldsymbol{x}', \boldsymbol{K}_f)$, denotes the state-dependent eigenvalues of the estimator's matrix $\boldsymbol{A}^*$



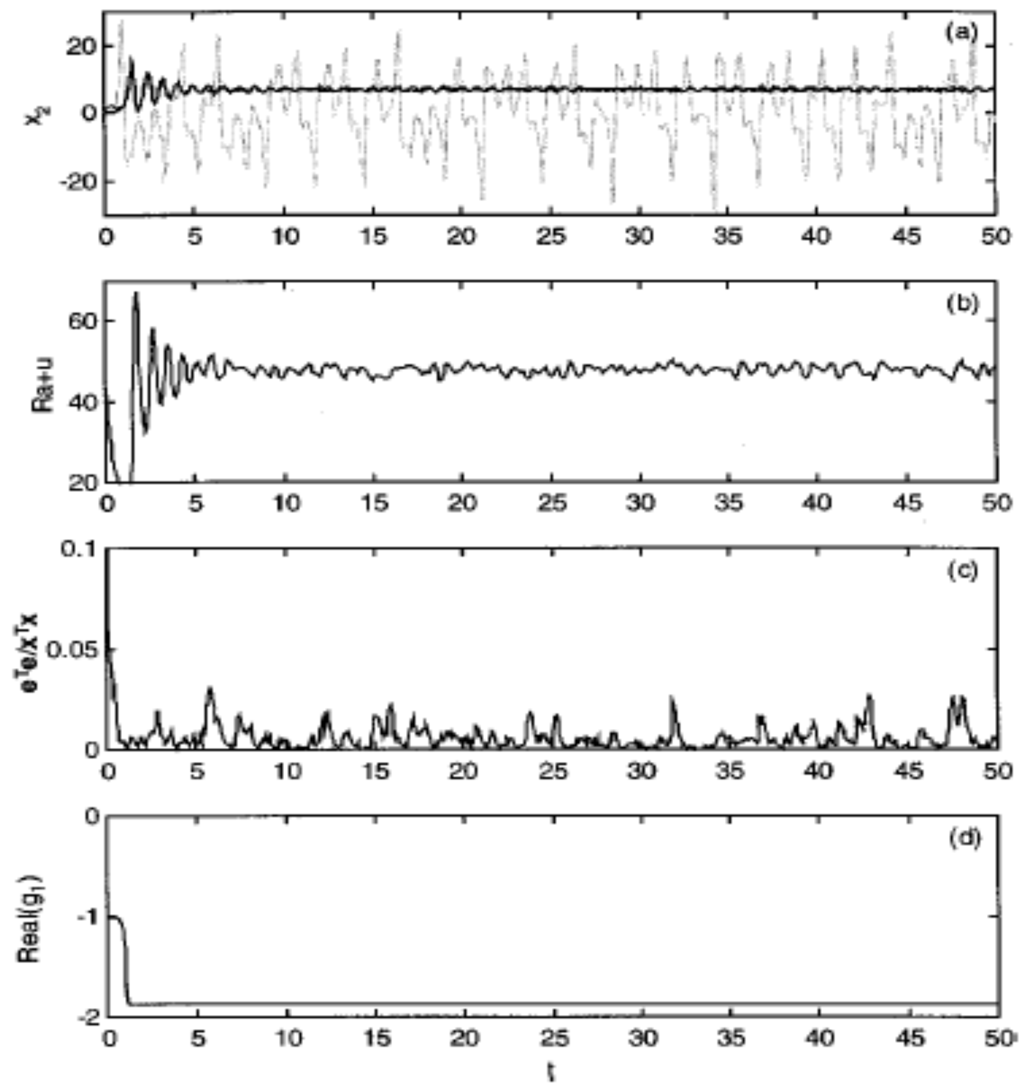
Linear optimal estimator

- Dropping the nonlinear term, we construct the estimator based on the linear system
- Comparing with nonlinear estimator, the operator ($\mathbf{A}$-$\mathbf{K}_f\mathbf{C}_i$) replaces the operator $\mathbf{A}^*$
- The optimal (Kalman) filter gain $\mathbf{K}_f = \mathbf{P}\mathbf{C}_i^T\mathbf{N}_i^{-1}$ that minimizes the error expectation is the solution of the matrix *Ricatti* equation

$$0 = \mathbf{AP} + \mathbf{PA}^T - \mathbf{PC}_i^T\mathbf{N}_i^{-1}\mathbf{C}_i\mathbf{P} + \mathbf{G}Q_\xi\mathbf{G}^T$$

THE CONTROLLER AND ESTIMATOR

The behavior of the optimally controlled nonlinear system with a nonlinear estimator. $\boldsymbol{K}_c=\{0.47, -0.54, 2.07\}$ and estimator $\boldsymbol{K}_f^T=\{4, 1.75, -0.35\}$. $Ra=3Ra_H=48$. One state variable is observed (x_2). As a function of time, the figure depicts (a) the temperature difference between positions 3 and 9 o'clock (x_2) (solid line), the estimate for x_2 (dashed line), and the behavior of the uncontrolled system (gray line); (b) the control signal, Ra+u; (c) the error, $\mathbf{e}^T\mathbf{e}/\mathbf{x}^T\mathbf{x}$; and (d) the largest real part of the state-dependent eigenvalues, $Real(g_1)$.



THE METHOD OF OTT-GREBOGI AND YORKE (OGY)

- The chaotic attractor consists of a very large number of non-stable periodic orbits
- Ott, Grebogi, and Yorke (1990) suggested a scheme dubbed *OGY* that encourages the chaotic system to follow one of the many unstable, periodic orbits through state space
- the chaotic system, with an appropriate control, can exhibit multiple behaviors.
- Advantage of OGY: the control can be affected empirically without knowledge of a model for the system.

Introduction to OGY method

- Consider a dynamics system $\frac{d\mathbf{x}}{dt} = F(\mathbf{x}, u)$
- *z(t)* is a measurable system variable
- Let $\boldsymbol{Z}(t)=\{z(t), z(t-\tau), z(t-2\tau), \ldots z(t-m\tau)\}$ (Packard et al., 1980, Takens, 1981). The vectors $\boldsymbol{Z}(t)$ are used to construct the phase space portrait of the attractor
- Then construct a Poincaré map of dimension *(m-1)*
- Periodic orbits will appear as either fixed points or a collection of discrete points through which the system's trajectories cycle

Implementation of the OGY method

- In the vicinity of the fixed point to be stabilized, one identifies the local stable and unstable manifolds
- Use the controller $u = -K_p \, \hat{\mathbf{n}} \bullet (\mathbf{Z_k} - \mathbf{Z^*})$ to nudge the trajectory towards the stable manifold

Kp is the controller's gain and **n** is a unit vector such that

$\hat{\mathbf{n}} \bullet \hat{e}_u = 1$ and $\hat{\mathbf{n}} \bullet \hat{e}_s = 0$

$\boldsymbol{e_u}$ and $\boldsymbol{e_s}$ are, respectively, unit vectors in the linear unstable and stable manifolds

Summary

- Various control strategies were used to alter the stability characteristics of the thermal convection loop – both in experiment and theory
- We examined
  - Ad-hoc proportional controller
  - Non-linear cubic controller to alter the direction of the bifurcation
  - Optimal (H_2) controller
- The thermal convection loop is a low-dimension system. Can similar techniques be applied to high-dimension systems?